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On the Validity of the Geometrical Theory of Diffraction by Star-Shaped Cylinders

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Submitted by C. L. Dolph

INTRODUCTION

Let $U(X, X_0; k)$ be the (Green's) function defined by the equations

$$(\Delta + k^2)\, U = \delta(X, X_0), \qquad X, X_0 \in \mathscr{E};$$
$$U = 0, \qquad X \in \mathscr{B}, \quad X_0 \in \mathscr{E};$$
$$\lim_{R\to\infty} \int_{|X|=R} |\, \partial U/\partial\, |X| - ikU\,|^2\, |\, dX\,| = 0,$$

where $\mathscr{E}$ is the (2-dimensional) region exterior to a piecewise smooth star-shaped curve $\mathscr{B}$. We obtain a rigorous asymptotic approximation of $U(X, X_0; k)$ in the shadow $S(X_0)$ of $\mathscr{B}$ under the assumption that $\mathscr{B}$ coincides with a circle $\mathscr{B}_0$ near the points of diffraction.

In Part I, using a priori estimates obtained by Morawetz and Ludwig [3], we establish that if $\mathscr{B}$ coincides with $\mathscr{B}_0$ in the shadow, then

$$U(X, X_0; k) = U_0(X, X_0; k)\, [1 + O(\exp\{-\, k^{1/3}\sigma\})]$$

as $k \to \infty$, uniformly on every closed bounded subset of $S(X_0)$. Here $U_0(X, X_0; k)$ is Green's function for the case $\mathscr{B} = \mathscr{B}_0$, and σ is a positive number independent of k and X.

Using this result, we prove in Part II that if $\mathscr{B}$ coincides with $\mathscr{B}_0$ only near the points of diffraction, then

$$U(X, X_0; k) = U_0(X, X_0; k)\, [1 + O(\exp\{-\, k^{1/3}\gamma\})]$$

as $k \to \infty$, uniformly on every closed bounded subset $S(X_0)$ sufficiently far from $\mathscr{B}$, where γ is a positive number independent of k and X.

The asymptotic approximations obtained in this study are believed to be the first ones established for solutions of diffraction problems with nonconvex boundaries. They were announced in [1]. These results generalize those

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obtained by Bloom and Matkowsky [2]; they considered diffraction by infinite cylinders of convex cross section. A reasonably complete account of other literature on this subject is given in [2].

PART I

Let $U_1(X, X_0; k)$ be the solution of the scattering problem P_1:

(i) $[\Delta + k^2]\, U = \delta(X, X_0)$, $X, X_0 \in \mathscr{E}_1$ (=exterior of closed curve $\mathscr{B}_1$);

(ii) $U = 0$, $X \in \mathscr{B}_1$ (=piecewise smooth and star-shaped);

(iii) $\lim_{R\to\infty} \int_{|X|=R} |\, \partial U/\partial \,|X| - ikU\,|^2 \cdot |\, dX\,| = 0.$

Assume: (1) $\mathscr{B}_1$ is obtained by deforming the portion of the circle $\mathscr{B}_0\,(=\{X : |\,X\,| = a\})$ "illuminated" by the "source" X_0 into a piecewise smooth arc A_1 (see Fig. 1).

(2) A_1 has no points on the tangents to $\mathscr{B}_0$ that pass through X_0.

(3) A_1 cuts $\mathscr{B}_0$ at a finite number of points.

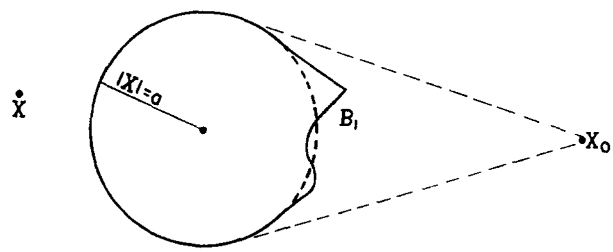

FIGURE 1

THEOREM 1. *As* $k \to \infty$,

$$U_1(X, X_0; k) = U_0(X, X_0; k) \cdot [1 + O(\exp\{-k^{1/3}\sigma\})],$$

uniformly in X *for* $X \in S_1^{<}(X_0)$.

Here σ is a positive number independent of X and k. The function $U_0(X, X_0; k)$ is the solution of the scattering problem P_0:

(i)′ $[\Delta + k^2]\, U = \delta(X, X_0)$, $|\,X\,|, |\,X_0\,| > a$;

(ii)′ $U = 0$; $|\,X\,| = a$;

(iii)′ $\lim_{R\to\infty} \int_{|X|=R} |\, \partial U/\partial \,|X| - ikU\,|^2 \cdot |\, dX\,| = 0.$

$S_1(X_0)$ is the "shadow" of $\mathscr{B}_1 : X \in S_1(X_0)$ if and only if $X \in \mathscr{E}_1 \cup \mathscr{B}_1$, and the straight line through X and X_0 cuts $\mathscr{B}_1$ at 2 distinct points $S_1{}^{<}(X_0)$ is any closed, bounded subset of $S_1(X_0)$ (see Fig. 2).

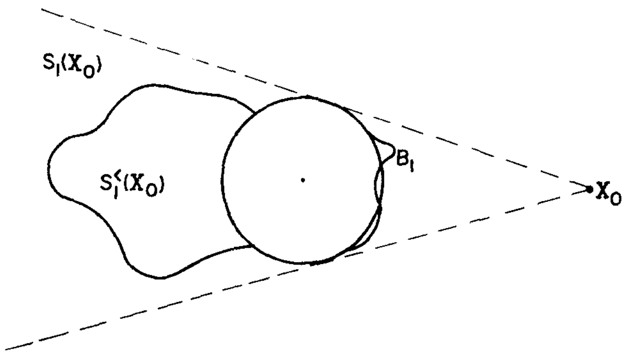

FIGURE 2

Applying Green's second identity to $U_1(X', X_0; k)$ and $U_0(X, X'; k)$, and then integrating over the region

$$\bar{\mathscr{E}}_1 \cap \bar{\mathscr{E}}_0 \qquad (\mathscr{E}_0 = \{X : | X | > a\}),$$

we get the integral equation

$$U_1(X, X_0; k) = U_0(X, X_0; k) + I_1(X, X_0; k) - I_2(X, X_0; k),$$

where

$$I_1(X, X_0; k) = \int_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_0} U_0(X, X'; k) \cdot \frac{\partial U_1}{\partial n}(X', X_0; k) \mid dX' \mid,$$

and

$$I_2(X, X_0; k) = \int_{\mathscr{S}} \frac{\partial U_0}{\partial n}(X, X'; k) \cdot U_1(X', X_0; k) \mid dX' \mid.$$

In the last integral

$$\mathscr{S} = \mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1 .$$

To prove Theorem 1 we show that as $k \to \infty$,

$$I_1(X, X_0; k) - I_2(X, X_0; k) = U_0(X, X_0; k) \cdot O(\exp\{- k^{1/3}\sigma\}),$$

uniformly in X for $X \in S_1{}^{<}(X_0)$. Setting

$$U_1(X', X_0; k) = U_1^{(s)}(X', X_0; k) + \frac{i}{4} H_0^{(1)}(k \mid X' - X_0 \mid),$$

($H_0^{(1)}(z)$ = Hankel function of first kind of order zero), we get the integral equation

$$I_1(X, X_0; k) = I_{11}(X, X_0; k) + I_{12}(X, X_0; k),$$

where

$$I_{11}(X, X_0; k) = \frac{i}{4} \int_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_0} U_0(X, X'; k) \cdot \frac{\partial H_0^{(1)}}{\partial n} (k \mid X' - X_0 \mid) \mid dX' \mid,$$

$$I_{12}(X, X_0; k) = \int_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_0} U_0(X, X'; k) \frac{\partial}{\partial n} U_1^{(s)}(X', X_0; k) \mid dX' \mid,$$

and the integral equation

$$I_2(X, X_0; k) = I_{21}(X, X_0; k) + I_{22}(X, X_0; k),$$

where

$$I_{21}(X, X_0; k) = \frac{i}{4} \int_{\mathscr{S}} \frac{\partial U_0}{\partial n} (X, X'; k) \cdot H_0^{(1)}(k \mid X' - X_0 \mid) \mid dX' \mid$$

$$I_{22}(X, X_0; k) = \int_{\mathscr{S}} \frac{\partial U_0}{\partial n} (X, X'; k) \cdot U_1^{(s)}(X', X_0; k) \mid dX' \mid.$$

Using Schwarz' inequality, we derive the estimate

$$\begin{aligned} I_1(X, X_0; k) \leqslant & \max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_0} \mid U_0(X, X'; k) \mid \\ & \cdot \left[L_1^{1/2} \left\{ \int_{\mathscr{B}_1} \left| \frac{\partial U_1^{(s)}}{\partial n} (X', X_0; k) \right|^2 \mid dX' \mid \right\}^{1/2} \right. \\ & \left. + \int_{\mathscr{B}_1} \left| \frac{\partial H_0^{(1)}}{\partial n} (k \mid X' - X_0 \mid) \right| \mid dX' \mid \right], \end{aligned}$$

where L_1 is the length of $\mathscr{B}_1$, and the estimate

$$\begin{aligned} I_2(X, X_0; k) \leqslant (2\pi a)^{1/2} & \max_{\mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right| \\ & \cdot \left[(2\pi a)^{1/2} \max_{\mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1} \mid U_1^{(s)}(X', X_0; k) \mid \right. \\ & \left. + \left\{ \int_{\mathscr{B}_0} \mid H_0^{(1)}(k \mid X' - X_0 \mid) \mid^2 \cdot \mid dX' \mid \right\}^{1/2} \right]. \end{aligned}$$

Morawetz and Ludwig [3] have obtained the following *a priori* estimates, as $k \to \infty$:

$$\text{(i)} \qquad \left\{ \left\{ \int_{\mathscr{B}_1} \left| \frac{\partial U_1^{(s)}}{\partial n} (X', X_0; k) \right|^2 \right\}^{1/2} \atop \max_{\mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1} | U_1^{(s)}(X', X_0; k)| \Big/ \left\{ \frac{\ln[ka/\rho]}{ka} \right\}^{1/2} \right\}$$

$$= O\left(\left\{ \int_{\mathscr{B}_1} | T \cdot \nabla H_0^{(1)}(k \mid X' - X_0 \mid)|^2 \mid dX' \mid \right\}^{1/2} + k \cdot \max_{\mathscr{B}_1} | H^{(1)}(k \mid X' - X_0 \mid)| \right)$$

$$= O(k^{1/2}),$$

uniformly in X_0, $X_0 \in$ (any closed subset of $\mathscr{E}_1$) where T is the unit tangent to $\mathscr{B}_1$ at X'.

$$\text{(ii)} \quad \int_{\mathscr{B}_1} \left| \frac{\partial H_0^{(1)}}{\partial n} (k \mid X' - X_0 \mid) \right| \cdot \mid dX' \mid = O(k^{1/2} / \mid X_0 \mid^{1/2}),$$

and

$$\text{(iii)} \quad \int_{\mathscr{B}_0} | H_0^{(1)}(k \mid X' - X_0 \mid)|^2 \cdot \mid dX' \mid = O\left(\frac{\ln k \mid X_0 \mid}{k \mid X_0 \mid} \right) + O\left(\frac{1}{k \mid X_0 \mid} \right),$$

uniformly in X_0 ($\mid X_0 \mid \geqslant \rho > 0$).

Consequently, as $k \to \infty$,

$$I_1(X, X_0; k) = \max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_0} | U_0(X, X'; k)| \cdot O(k^{1/2}),$$

$$I_2(X, X_0; k) = \max_{\mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right| \cdot O(\{\ln[ka/\rho]\}^{1/2}),$$

uniformly in X_0 for $X_0 \in$ (any closed subset of $\mathscr{E}_1$).

As $k \to \infty$,

$$U_0^{-1}(X, X_0; k) = O(k^{1/2} \cdot \exp\{k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3} \lambda_{<}(X, X_0)\}),$$

uniformly in X for $X \in S_0^{<}(X_0)$ ($= S_1^{<}(X_0)$);

$$U_0(X, X'; k) = \sum_1^2 O(k^{-1/2} \cdot \exp\{- k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3} \lambda_m(X, X')\}),$$

$$\frac{\partial U_0}{\partial n} (X, X'; k) = \sum_1 O(k^{1/2} \cdot \exp\{- k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3} \lambda_m(X, X')\}),$$

uniformly in X' and X for $X' \in [\mathscr{B}_1 \cap \bar{\mathscr{E}}_0] \cup [\mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1]$ and $X \in S_1{}^{<}(X')$ $(\subseteq S_1{}^{<}(X_0))$.

Here $2^{1/3}e^{2\pi i/3}\tau_1$ is that zero of the Airy function closest to zero, and

$$\lambda_<(X, X_0) = \min_{m=1,2} \lambda_m(X, X_0),$$

$$\lambda_1(X, X') = a[|\,\theta - \theta'\,| - \arccos[a/|\,X\,|] - \arccos[a/|\,X'\,|]],$$

$$\lambda_2(X, X') = a[2\pi - |\,\theta - \theta'\,| - \arccos[a/|\,X\,|] - \arccos[a/|\,X'\,|]],$$

$$\theta = \arg X, \qquad \text{and} \qquad \theta' = \arg X'.$$

Let $\hat{X}_m$ be the point on

$$\mathscr{T}_1 = [\mathscr{B}_1 \cap \bar{\mathscr{E}}_0] \cup [\mathscr{B}_0 \cap \bar{\mathscr{E}}_1 - \mathscr{B}_0 \cap \mathscr{B}_1]$$

such that

$$\lambda_m(X, \hat{X}_m) = \min_{X' \in \mathscr{T}_1} \lambda_m(X, X'), \; X \in S_1{}^{<}(X_0).$$

If $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_1(X, X_0)$, $\theta_0 = 0$,

(ii) $\lambda_1(X, \hat{X}_1) = \lambda_1(X, X_0) + \mu_1(X)$, $\mu_1(X) > 0$, and

(iii) $\mu_1(X) \geqslant \min_{S_1{}^{<}(X_0)} \mu_1(X') \geqslant \varDelta > 0$.

Statements (i)–(iii) imply the inequality

$$-\lambda_1(X, \hat{X}_1) \leqslant -\lambda_<(X, X_0) - \varDelta \qquad \text{(see Fig. 3).}$$

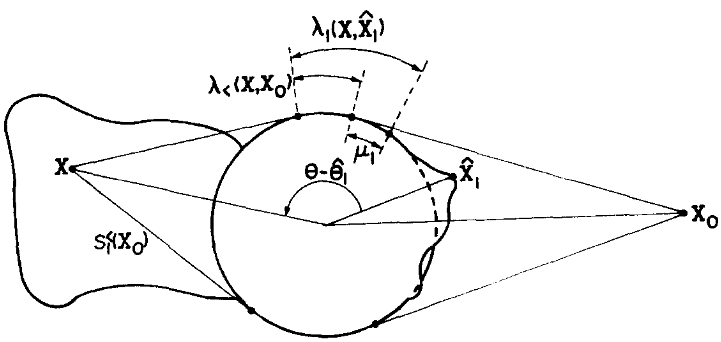

FIGURE 3

If $0 \leqslant \theta \leqslant \pi$ and $0 \geqslant \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_1(X, X_0)$, $\theta_0 = 0$,
(ii) $\lambda_2(X, X_0) \geqslant \lambda_1(X, X_0)$, $\theta_0 = 0$,
(iii) $\lambda_2(X, X_0)|_{\theta_0=0} = \lambda_1(X, X_0)|_{\theta_0=2\pi}$,
(iv) $\lambda_1(X, \hat{X}_1) = \lambda_1(X, X_0)|_{\theta_0=2\pi} + \mu_2(X)$, $\mu_2(X) > 0$,
(v) $\mu_2(X) \geqslant \min_{S_1^<(X_0)} \mu_2(X') \geqslant \Delta > 0$.

Statements (i)–(v) imply the inequality

$$-\lambda_1(X, \hat{X}_1) \leqslant -\lambda_<(X, X_0) - \Delta \qquad \text{(see Fig. 4).}$$

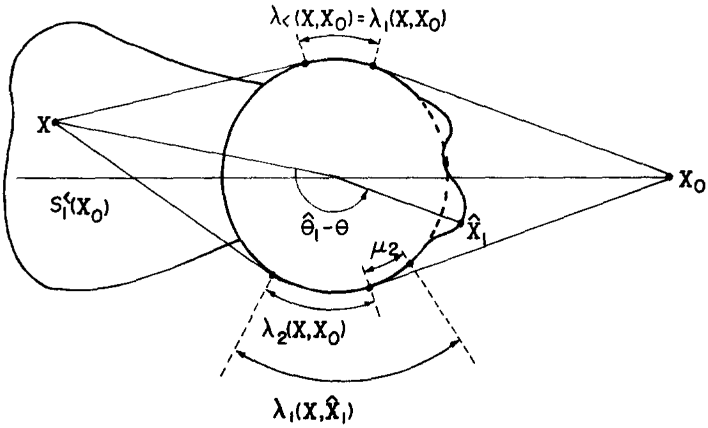

FIGURE 4

If $\pi \leqslant \theta \leqslant 2\pi$ and $0 \geqslant \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_2(X, X_0)$, $\theta_0 = 0$,
(ii) $\lambda_2(X, X_0)|_{\theta_0=0} = \lambda_1(X, X_0)|_{\theta_0=2\pi}$,
(iii) $\lambda_1(X, \hat{X}_1) = \lambda_1(X, X_0)|_{\theta_0=2\pi} + \mu_3(X)$, $\mu_3(X) > 0$,
(iv) $\mu_3(X) \geqslant \min_{S_1^<(X_0)} \mu_3(X') \geqslant \Delta > 0$.

Statements (i)–(iv) imply the inequality

$$-\lambda_1(X, \hat{X}_1) \leqslant -\lambda_<(X, X_0) - \Delta \qquad \text{(see Fig. 5).}$$

If $\pi \leqslant \theta \leqslant 2\pi$ and $0 \leqslant \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_2(X, X_0)$, $\theta_0 = 0$,
(ii) $\lambda_2(X, X_0) \leqslant \lambda_1(X, X_0)$, $\theta_0 = 0$,
(iii) $\lambda_1(X, \hat{X}_1) = \lambda_1(X, X_0)|_{\theta_0=0} + \mu_4(X)$, $\mu_4(X) > 0$,
(iv) $\mu_4(X) \geqslant \min_{S_1^<(X_0)} \mu_4(X') \geqslant \Delta > 0$.

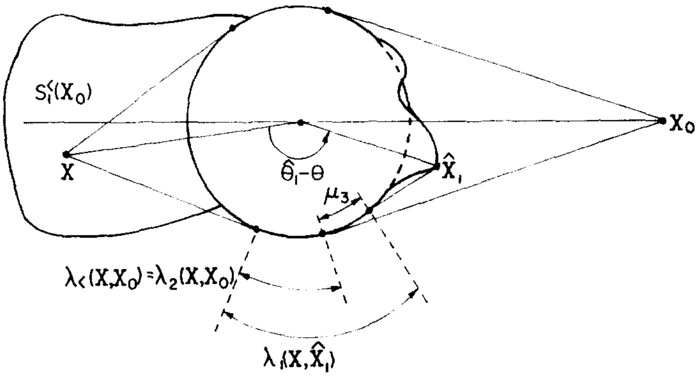

FIGURE 5

Statements (i)–(iv) imply the inequality

$$-\lambda_1(X, \hat{X}_1) \leqslant -\lambda_<(X, X_0) - \Delta \qquad \text{(see Fig. 6).}$$

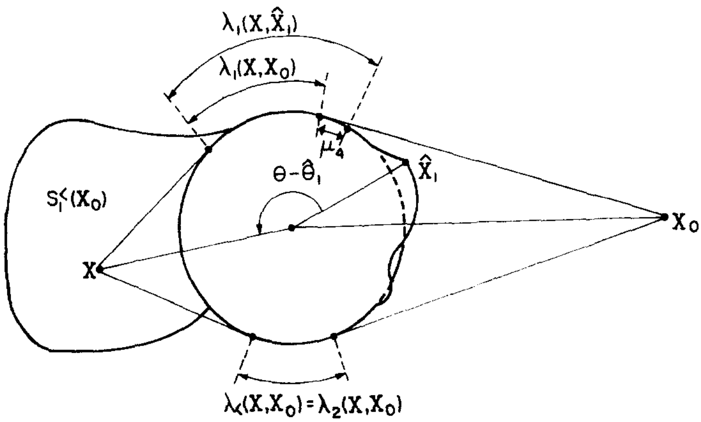

FIGURE 6

It follows from the above series of inequalities that for all $X \in S_1{}^<(X_0)$

$$-\lambda_1(X, \hat{X}_1) \leqslant -\lambda_<(X, X_0) - \Delta,$$

where Δ is positive and independent of X.

Similarly,

$$-\lambda_2(X, \hat{X}_2) \leqslant -\lambda_<(X, X_0) - \Delta$$

for all $X \in S_1^<(X_0)$.

Since

$$\left.\begin{array}{r} -\min\limits_{\mathscr{B}_1\cap\bar{\mathscr{E}}_0} \lambda_m(X, X') \\ -\min\limits_{\mathscr{B}_0\cap\bar{\mathscr{E}}_1-\mathscr{B}_0\cap\mathscr{B}_1} \lambda_m(X, X') \end{array}\right\} \leqslant -\lambda_m(X, \hat{X}_m),$$

it follows from the above estimates for $U_0(X, X'; k)$, $\partial U_0(X, X'; k)/\partial n$, and $U_0^{-1}(X, X_0; k)$, that as $k \to \infty$,

$$\max_{\mathscr{B}_1\cap\bar{\mathscr{E}}_0} | U_0(X, X'; k)|$$
$$= | U_0(X, X_0; k)| \cdot O(\exp\{- k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3}\Delta\}),$$

and

$$\max_{\mathscr{B}_0\cap\bar{\mathscr{E}}_1-\mathscr{B}_0\cap\mathscr{B}_1} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right|$$
$$= | U_0(X, X_0; k)| \cdot O(k^{1/2} \exp\{- k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3}\Delta\}),$$

uniformly in X for $X \in S_1{}^{<}(X_0)$.

We therefore conclude that as $k \to \infty$,

$$I_1(X, X_0; k) = \max_{\mathscr{B}_1\cap\bar{\mathscr{E}}_0} | U_0(X, X'; k)| \cdot O(k^{1/2})$$
$$= | U_0(X, X_0; k)| \cdot O(\exp\{- k^{1/3}\sigma\}),$$

and

$$I_2(X, X_0; k) = \max_{\mathscr{B}_0\cap\bar{\mathscr{E}}_1-\mathscr{B}_0\cap\mathscr{B}_1} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right| \cdot O(\{\ln[ka/\rho]\}^{1/2})$$
$$= | U_0(X, X_0; k)| \cdot O(\exp\{- k^{1/3}\sigma\}),$$

uniformly in X for $X \in S_1{}^{<}(X_0)$.

Part II

Let $U_2(X, X_0; k)$ be the solution of the scattering problem P_2:

(i) $[\Delta + k^2]\, U = \delta(X, X_0)$, $X, X_0 \in \mathscr{E}_2$ (= exterior of closed curve $\mathscr{B}_2$);

(ii) $U = 0$, $X \in \mathscr{B}_2$ (= star-shaped deformation of $\mathscr{B}_1$);

(iii) $\lim\limits_{R\to\infty} \int_{|X|=R} | \partial U/\partial \,| X | - ikU |^2 \cdot | dX | = 0.$

Assume: (1) $\mathscr{B}_2$ is obtained by deforming the "dark" portion of $\mathscr{B}_1$ into a piecewise smooth arc A_2; see Fig. 7. (2) A_2 cuts $\mathscr{B}_1$ at a finite number of points.

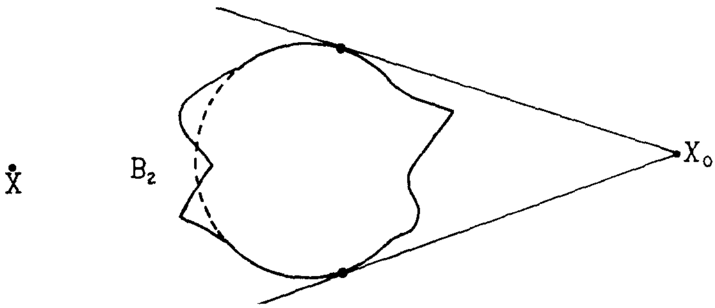

FIGURE 7

THEOREM 2. *If $\mathscr{B}_2$ is obtained by deforming the "dark" portion of the boundary of Theorem* I *into a piecewise smooth arc A_2, then as $k \to \infty$*

$$U(X, X_0; k) = U_0(X, X_0; k) \cdot [1 + O(\exp\{-k^{1/3}\sigma\})],$$

uniformly in X, for $X \in S_2^<(X_0) - \mathscr{R}$.

Here $S_2^<(X_0)$ is any bounded closed subset of the shadow $S_2(X_0)$ of $\mathscr{B}_2$, and $\mathscr{R}$ is the "region of influence" of A_2 defined as follows. Let e be the end of A_2 closest to the illuminated side of $\mathscr{B}_2$, and let f be a point on $\mathscr{B}_2$ between e, and the shadow boundary at g. Let f' be a point on $\mathscr{B}_2$ between the other end of A_2 and the shadow boundary, that is as far from the illuminated side of $\mathscr{B}_2$ as f. Assume f is so located that the tangents to $\mathscr{B}_2$ at f and f' have no points on A_2. $\mathscr{R}$ is that part of $S_2(X_0)$ bounded by $\mathscr{B}_2$, and the tangents from f and f' (see Fig. 8).

Applying Green's second identity to $U_2(X, X'; k)$ and $U_1(X', X_0; k)$, and then integrating over the region $\bar{\mathscr{E}}_2 \cap \bar{\mathscr{E}}_1$, we get the integral equation

$$U_2(X, X_0; k) = U_1(X, X_0; k) - I_3(X, X_0; k) + I_4(X, X_0; k),$$

where

$$I_3(X, X_0; k) = \int_{\mathscr{B}_2 \cap \bar{\mathscr{E}}_1} U_1(X', X_0; k) \cdot \frac{\partial U_2}{\partial n}(X, X'; k) \cdot |\, dX' \,|$$

and

$$I_4(X, X_0; k) = \int_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_2 \cap \mathscr{B}_1} \frac{\partial U_1}{\partial n}(X', X_0; k) \cdot U_2(X, X'; k) \cdot |\, dX' \,| .$$

To prove Theorem 2 we show that as $k \to \infty$,

$$I_3(X, X_0; k) - I_4(X, X_0; k) = U_1(X, X_0; k) \cdot O(\exp\{-k^{1/3}\gamma\}),$$

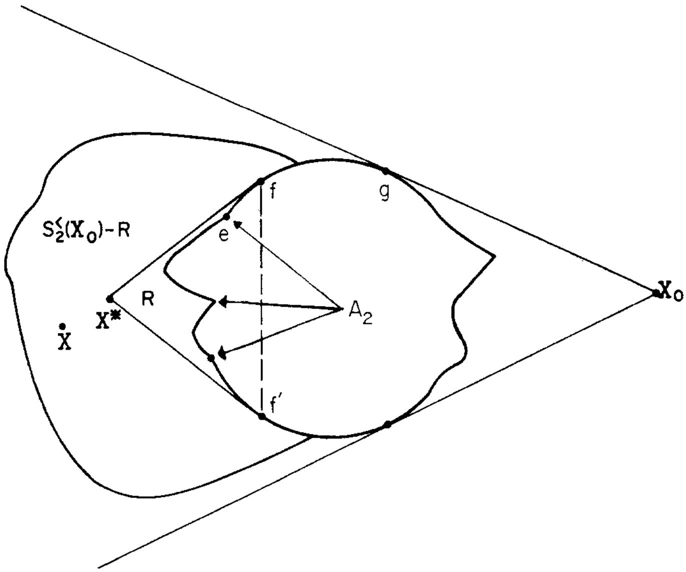

FIGURE 8

uniformly in X for $X \in S_2{}^{<}(X_0) - \mathscr{R}$. Set

$$U_2(X, X'; k) = U_2^{(s)}(X, X'; k) + \frac{i}{4} H_0^{(1)}(k \mid X - X' \mid)$$

in the above integrals and use Schwarz' inequality to make the estimates

$$\begin{aligned} | I_3(X, X_0; k)| \leqslant \max_{\mathscr{B}_2 \cap \bar{\mathscr{E}}_1} & | U_1(X', X_0; k)| \\ & \cdot \left[L_2^{1/2} \left\{ \int_{\mathscr{B}_2} \left| \frac{\partial U_2^{(s)}}{\partial n} (X, X'; k) \right|^2 | dX' | \right\}^{1/2} \right. \\ & \left. + \int_{\mathscr{B}_2} \left| \frac{\partial H_0^{(1)}}{\partial n} (k \mid X - X' \mid) \right| | dX' | \right], \end{aligned}$$

and

$$\begin{aligned} | I_4(X, X_0; k)| \leqslant L_1^{1/2} \cdot & \max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_1 \cap \mathscr{B}_2} \left| \frac{\partial U_1}{\partial n} (X', X_0; k) \right| \\ & \cdot \left[L_1^{1/2} \cdot \max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_1 \cap \mathscr{B}_2} | U_2^{(s)}(X, X'; k)| \right. \\ & \left. + \left\{ \int_{\mathscr{B}_1} | H_0^{(1)}(k \mid X - X' \mid)|^2 \cdot | dX' | \right\} \right], \end{aligned}$$

where L_2 is the length of $\mathscr{B}_2$.

By essentially the same argument used to prove Theorem 1 we get the result that as $k \to \infty$

$$\frac{\partial U_1}{\partial n}(X', X_0; k) = \frac{\partial U_0}{\partial n}(X', X_0; k) \cdot [1 + O(\exp\{-k^{1/3}\sigma\})],$$

uniformly in X' for $X' \in S_2^<(X_0)$ $(\subseteq S_1^<(X_0))$.

It follows from this result, Theorem I, the estimates of Morawetz and Ludwig [3], and the above inequalities that, as $k \to \infty$,

$$I_3(X, X_0; k) = \max_{\mathscr{B}_2 \cap \bar{\mathscr{E}}_1} | U_0(X', X_0; k)| \cdot O(k^{1/2}),$$

$$I_4(X, X_0; k) = \max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_1 \cap \mathscr{B}_2} \left| \frac{\partial U_0}{\partial n}(X', X_0; k) \right| \cdot O(\{\ln[ka/\rho]\}^{1/2}),$$

uniformly in X for $X \in S_2^<(X_0) - \mathscr{R}$.

It follows from Theorem 1 that, as $k \to \infty$,

$$U_1^{-1}(X, X_0; k) = O(k^{1/2} \exp\{+ k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3} \lambda_<(X, X_0)\})$$

uniformly in X for $X \in S_2^<(X_0) - \mathscr{R}$ $(\subseteq S_1^<(X_0))$.

Furthermore, as $k \to \infty$

$$\max_{\mathscr{B}_2 \cap \bar{\mathscr{E}}_1} | U_0(X', X_0; k)|$$
$$= \sum_{m=1}^{2} O(k^{-1/2} \exp\{- k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3} \lambda_m(\tilde{X}_m, X_0)\}),$$

and

$$\max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_1 \cap \mathscr{B}_2} \left| \frac{\partial U_0}{\partial n}(X', X_0; k) \right|$$
$$= \sum_{1}^{2} O(k^{1/2} \exp\{- k^{1/3}(\operatorname{Im} \tau_1)\, a^{-2/3} \lambda_m(\tilde{X}_m, X_0)\}),$$

where

$$\lambda_m(\tilde{X}_m, X_0) = \min_{X' \in \mathscr{T}_2} \lambda_m(X', X_0),$$

with

$$\mathscr{T}_2 = [\mathscr{B}_2 \cap \bar{\mathscr{E}}_1] \cup [\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_1 \cap \mathscr{B}_2].$$

To complete the proof of Theorem 2 we show that if $X \in S_2^<(X_0) - \mathscr{R}$ then

$$-\lambda_m(\tilde{X}_m, X_0) \leqslant -\lambda_<(X, X_0) - \Lambda,$$

where Λ is positive and independent of X. It then follows from the above estimates that, as $k \to \infty$,

$$\begin{aligned} I_3(X, X_0; k) &= \max_{\mathscr{B}_2 \cap \bar{\mathscr{E}}_1} | U_0(X', X_0; k)| \cdot O(k^{1/2}) \\ &= | U_1(X, X_0; k)| \cdot O(\exp\{- k^{1/3}\gamma\}), \\ I_4(X, X_0; k) &= \max_{\mathscr{B}_1 \cap \bar{\mathscr{E}}_2 - \mathscr{B}_1 \cap \mathscr{B}_2} \left| \frac{\partial U_0}{\partial n}(X', X_0; k) \right| \cdot O(\{\ln[ka/p]\}^{1/2}) \\ &= | U_1(X, X_0; k)| \cdot O(\exp\{- k^{1/3}\gamma\}), \end{aligned}$$

uniformly in X, $X \in S_2^{<}(X_0) - \mathscr{R}$.

Now, if $0 \leqslant \theta \leqslant \pi$, and $X \in S_2^{<}(X_0) - \mathscr{R}$, then

(i) $\lambda_<(X, X_0) = \lambda_1(X, X_0)$, $\theta_0 = 0$,

(ii) $\lambda_1(\tilde{X}_1, X_0) = \lambda_1(X, X_0) + \nu_1(X)$, $\nu_1(X) > 0$,

(iii) $\nu_1(X) \geqslant \min_{S_2^{<}(X_0) - \mathscr{R}} \nu_1(X') \geqslant \Lambda > 0$.

Statements (i)–(iii) imply the inequality

$$-\lambda_1(\tilde{X}_1, X_0) \leqslant -\lambda_<(X, X_0) - \Lambda \qquad \text{(see Fig. 9)}.$$

If $\pi \leqslant \theta \leqslant 2\pi$, and $X \in S_2^{<}(X_0) - \mathscr{R}$, then

(i) $\lambda_<(X, X_0) = \lambda_2(X, X_0)$, $\theta_0 = 0$,

(ii) $\lambda_1(X^*, X_0)|_{\theta_0 = 0} = \lambda_2(X^*, X_0)|_{\theta_0 = 2\pi}$,

(iii) $\lambda_1(\tilde{X}_1, X_0) = \lambda_1(X^*, X_0)|_{\theta_0 = 0} + \nu_2$, $\nu_2 > 0$,

(iv) $\lambda_2(X^*, X_0)|_{\theta_0 = 2\pi} \geqslant \lambda_2(X, X_0)|_{\theta_0 = 0}$,

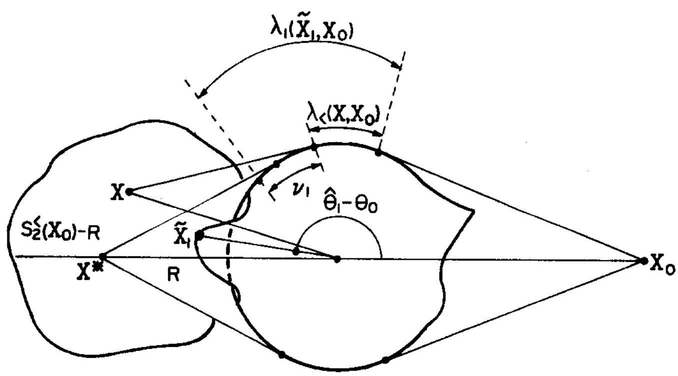

FIGURE 9

Statements (i)–(iv) imply the inequality

$$-\lambda_1(\tilde{X}_1, X_0) \leqslant -\lambda_<(X, X_0) - \Lambda \qquad \text{(see Fig. 10).}$$

It follows from the above series of inequalities that, for all $X \in S_2{}^<(X_0) - \mathscr{R}$,

$$-\lambda_1(\tilde{X}_1, X_0) \leqslant -\lambda_<(X, X_0) - \Lambda,$$

and similarly that

$$-\lambda_2(\tilde{X}_2, X_0) \leqslant -\lambda_<(X, X_0) - \Lambda.$$

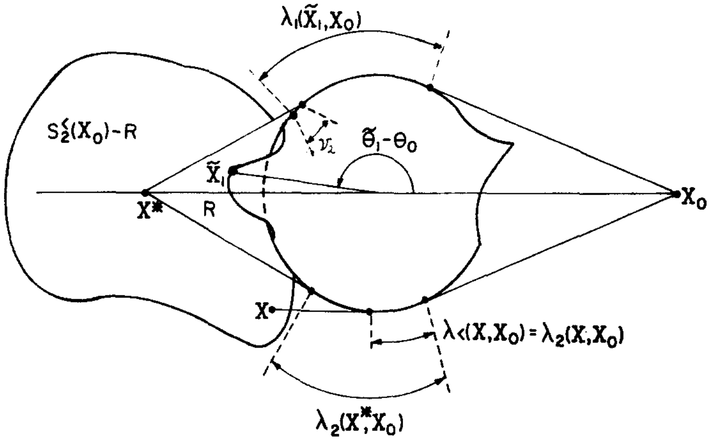

FIGURE 10

CONCLUSION

In Theorems 1 and 2 we could just as well let $\mathscr{B}_0$ be any convex curve such that $U_0(X, X_0; k)$ behaves asymptotically as predicted by the geometrical theory of diffraction.

Furthermore, asymptotic approximations similar to the above can be obtained in 3-dimensions for perturbed spheres or for star-shaped perturbations of any convex surface S for which the geometrical theory of diffraction is known to be valid.

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